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On the Non-negativity of the First Coefficient of Kazhdan–Lusztig Polynomials

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INTRODUCTION

The purpose of this article is to show the non-negativity of the first coefficient of Kazhdan–Lusztig polynomials for arbitrary Coxeter systems.

Let (W, S) be an arbitrary Coxeter system, and let $P_{x,w} = \sum_{i \geq 0} p_i(x,w)q^i \in \mathbb{Z}[q]$ be the Kazhdan–Lusztig polynomial for $x, w \in W$. In Section 1, we recall the definition of Kazhdan–Lusztig polynomials and that of left subwords. Also, we show some lemmas with respect to left subwords. In Section 2, we prove the following: for $x, w \in W$ with $x \leq w$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w and let $x = s_1 s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m$ be a left subword of $s_1 s_2 \cdots s_m$, where $s_r \widehat{s_{r+1}} s_{r+2}$ means $s_r s_{r+2}$. For each $j \in \{0, 1, 2, \ldots, k\}$, we put $w_j := s_1 s_2 \cdots \widehat{s_{i_{j+1}}} \cdots \widehat{s_{i_{j+2}}} \cdots \widehat{s_{i_k}} \cdots s_m$. Then, we have $x = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{k-1} \lessdot w_k = w$ and $p_1(x,w) \geq p_1(w_1,w) \geq p_1(w_2,w) \geq \cdots \geq p_1(w_{k-1},w) \geq p_1(w,w) = 0$, where $y \lessdot z$ means that $y < z$ in the Bruhat order and $l(y) = l(z) - 1$ (l is the length function).

Let us give a brief review of known results. It is conjectured in [8] that all coefficients of Kazhdan–Lusztig polynomials are non-negative. This is still an open problem, but some of special cases are verified. The conjecture is correct for finite Coxeter groups, affine Weyl groups and crystallographic Coxeter groups (cf. [1, 9, 6]). Dyer [5] proved the non-negativity of $p_1(e,w)$ for $w \in W$, where e is the identity element. In [11] we found a combinatorial description for $p_1(x,w)$ (cf. theorem [11, Theorem A] in this article) and proved the non-negativity of $p_1(x,w)$ for $x, w \in W$ satisfying $l(w) = l(x) + l(x^{-1}w)$ or $l(w) = l(x) + l(wx^{-1})$.

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1. PRELIMINARIES AND NOTATIONS

At first, we will define the notion of the Bruhat order and Kazhdan–Lusztig polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, where S denotes a privileged set of involutions in W. The standard references are [3, 7] for the Bruhat order and Kazhdan–Lusztig polynomials.

DEFINITION 1.1 (Bruhat order). We put $T := \{wsw^{-1};\ s \in S,\ w \in W\}$. For $y, z \in W$, we denote $y <' z$ if and only if there exists an element t of T such that $l(tz) < l(z)$ and $y = tz$, where l is the length function. Then the Bruhat order denoted by $\leq$ is defined as follows. For $x, w \in W$, $x \leq w$ if and only if there exists a sequence $x_0, x_1, \ldots, x_r$ in W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$. We also use the notation $x \lessdot w$ if $x < w$ and $l(x) = l(w) - 1$.

The following is well known (cf. [7]). For $w \in W$, let $w = s_1 s_2 \cdots s_m$ be a reduced decomposition of w (i.e., all $s_i \in S$ and $l(w) = m$). For $x \in W$, $x \leq w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \ldots, i_t$ such that $1 \leq i_1 < i_2 < \cdots < i_t \leq m$ and $x = s_{i_1} s_{i_2} \cdots s_{i_t}$. This expression is not reduced, in general; i.e., it may be the case that $l(x) < t$. However, it is known that one can find a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq m$, $x = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $l(x) = k$.

Also, the following decomposition is well known (cf. [7]). Let J be a subset of S. We put $W_J :=$ subgroup of W generated by J and $W^J := \{y \in W;\ l(yz) = l(y) + l(z)$ for any $z \in W_J\}$. Then, for $w \in W$, there exist an element w^J of W^J and an element w_J of W_J such that $w = w^J w_J$.

DEFINITION 1.2 (Kazhdan–Lusztig polynomial). For $x, w \in W$, we define the Kazhdan–Lusztig polynomial for x, w denoted by $P_{x,w} = \sum_{i \geq 0} p_i(x, w) q^i \in \mathbb{Z}[q]$ as follows: $P_{x,w} = 0$ if $x \not\leq w$, $P_{x,w} = 1$ if $x = w$. When $x < w$, for fixed $s \in S$ satisfying $l(sw) < l(w)$, we set $c := 0$ if $x \lessdot sx$ and $c := 1$ if $sx \lessdot x$. Then, $P_{x,w}$ is defined inductively as

$$P_{x,w} = q^{1-c} P_{sx,sw} + q^c P_{x,sw} - \sum_{sz < z < sw} \mu(z, sw) q^{(l(w) - l(z))/2} P_{x,z},$$

where $\mu(z, sw)$ is the coefficient of $q^{(l(sw) - l(z) - 1)/2}$ of $P_{z,sw}$.

Remark. For the equivalence of this definition with the original definition in [8], we refer the reader to [7].

Next, we define the notion of left subwords and we state some properties.

DEFINITION 1.3 (Left subword). For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w. We put $[m] := \{1, 2, \ldots, m\}$ and define a map f from $[m]$ to S by $f(i) = s_i$. Let $J = (j_1, j_2, \ldots, j_r)$ and $J' = (j'_1, j'_2, \ldots, j'_r)$ be subsequences of $(1, 2, \ldots, m)$. We call $f|_J$ a subword of $s_1 s_2 \cdots s_m$ and we set $|f|_J| = f(j_1)f(j_2)\cdots f(j_r) = s_{j_1}s_{j_2}\cdots s_{j_r}$, where $f|_J$ denotes the restriction of f to J. We define $J < J'$ if and only if there exists a natural number k such that $k \le r$, $j_1 = j'_1$, $j_2 = j'_2, \ldots, j_{k-1} = j'_{k-1}$ and $j_k < j'_k$ (i.e., lexicographic order). For $x \in W$ with $x \le w$ and $l(x) = r$, if

$$\begin{aligned} I &= (i_1, i_2, \ldots, i_r) \\ &= \min\{\text{a subsequence } J = (j_1, j_2, \ldots, j_r) \text{ of } (1, 2, \ldots, m);\ |f|_J| = x\} \end{aligned}$$

then we call $f|_I$ a left subword of $s_1 s_2 \cdots s_m$.

From now on, we identify $|f|_J|$ with $f|_J$ when there is no confusion.

Remark. For $x, w \in \dot{W}$ with $x \le w$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w. When $x = s_1 s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a left subword of $s_1 s_2 \cdots s_m$, $s_m s_{m-1} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_{k-1}}} \cdots \widehat{s_{i_1}} \cdots s_m$ equals $\mathfrak{D}_0(x^{-1})$, which is the special one of distinguished subexpressions for $s_m s_{m-1} \cdots s_1$ introduced in [4].

LEMMA 1.4. *For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w and let $l(s_1 s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m) = m - k$, where $s_r \widehat{s_{r+1}} s_{r+2}$ means $s_r s_{r+2}$. Then, the following conditions are equivalent*:

(1) *$s_1 s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a left subword of $s_1 s_2 \cdots s_m$.*

(2) *For each $j \in [k]$, $s_{i_j} s_{i_j + 1} \cdots \widehat{s_{i_{j+1}}} \cdots \widehat{s_{i_{j+2}}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a reduced decomposition.*

(3) *For each $j \in [k]$, $s_1 s_2 \cdots \widehat{s_{i_{j+1}}} \cdots \widehat{s_{i_{j+2}}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a left subword of $s_1 s_2 \cdots s_m$.*

Proof. By the definition of left subwords, we can easily check (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (2). So, we will show (1) $\Rightarrow$ (3). Noting the statements (1) and (3), it suffices to prove that $s_1 s_2 \cdots s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a reduced decomposition. We suppose that $s_1 s_2 \cdots s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m$ is not a reduced decomposition and find a contradiction. By (1) $\Leftrightarrow$ (2), there exists a natural number a such that $2 \le a \le i_1$, $s_a s_{a+1} \cdots s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a reduced decomposition, and $s_{a-1} s_a s_{a+1} \cdots s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m$ is not a reduced decomposition. Since $s_a s_{a+1} \cdots s_{i_2 - 1}$ is a reduced decomposition, there exists a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $i_2 = j_1 < j_2 < \cdots < j_k \le m$ and

$$s_a s_{a+1} \cdots s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m = s_{a-1} s_a \cdots s_{i_1} \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_2}} \cdots \widehat{s_{j_k}} \cdots s_m .$$

Hence, we have

$$s_a \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m < s_{a-1}s_a \cdots s_{i_1} \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_2}} \cdots \widehat{s_{j_k}} \cdots s_m.$$

Since $s_{a-1}s_a \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a reduced decomposition, we see

$$s_a \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_k}} \cdots s_m = s_a \cdots s_{i_1} \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_2}} \cdots \widehat{s_{j_k}} \cdots s_m.$$

Therefore, we obtain

$$s_1s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m = s_1s_2 \cdots s_{i_1} \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_2}} \cdots \widehat{s_{j_k}} \cdots s_m.$$

But this contradicts that $s_1s_2 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_k}} \cdots s_m$ is a left subword of $s_1s_2 \cdots s_m$. It follows that we have (1) $\Rightarrow$ (3). ∎

LEMMA 1.5. *For $x, w \in W$, let $s_1s_2 \cdots s_m$ be a reduced decomposition of w and let $x = s_{i_1}s_{i_2} \cdots s_{i_k}$ be a left subword of $s_1s_2 \cdots s_m$. Then, we have the following*:

(1) *For any $q \in [k]$, $s_{i_1}s_{i_2} \cdots s_{i_q}$ is a left subword of $s_1s_2 \cdots s_m$.*

(2) *For $p < i_k$, we have $x \not\leq s_1s_2 \cdots s_p$.*

(3) *If there exists an element s of S such that $x = s_{i_1}s_{i_2} \cdots \widehat{s_{i_d}} \cdots s_{i_k}s$, then $s_{i_1}s_{i_2} \cdots \widehat{s_{i_d}} \cdots s_{i_k}$ is a left subword of $s_1s_2 \cdots s_m$.*

Proof. (1) is trivial by Lemma 1.4. (2) We suppose that there exists a natural number p such that $p < i_k$ and $x \leq s_1s_2 \cdots s_p$. Let $x = s_{j_1}s_{j_2} \cdots s_{j_k}$ be a left subword of $s_1s_2 \cdots s_p$. Then, by Lemma 1.4, $s_{j_1}s_{j_2} \cdots s_{j_k}$ is also a left subword of $s_1s_2 \cdots s_m$. Since $j_k \leq p < i_k$, this contradicts the definition of left subwords. (3) We assume that $s_{i_1}s_{i_2} \cdots \widehat{s_{i_d}} \cdots s_{i_k}$ is not a left subword of $s_1s_2 \cdots s_m$. By Lemma 1.4, there exist natural numbers a and b such that $i_b < a < i_{b+1}$, $b + 1 \leq d$ and $s_as_{i_{b+1}}s_{i_{b+2}} \cdots \widehat{s_{i_d}} \cdots s_{i_k}$ is not a reduced decomposition; here we put $i_0 := 0$. Thus, $s_as_{i_{b+1}}s_{i_{b+2}} \cdots \widehat{s_{i_d}} \cdots s_{i_k}s$ is not a reduced decomposition. Hence, by the equality $s_{i_{b+1}}s_{i_{b+2}} \cdots \widehat{s_{i_d}} \cdots s_{i_k}s = s_{i_{b+1}}s_{i_{b+2}} \cdots s_{i_k}$, $s_as_{i_{b+1}}s_{i_{b+2}} \cdots s_{i_k}$ is not a reduced decomposition. This contradicts that $s_{i_1}s_{i_2} \cdots s_{i_k}$ is a left subword of $s_1s_2 \cdots s_m$. ∎

LEMMA 1.6. *Let $s_{i_1}s_{i_2} \cdots s_{i_k}$ be a left subword of a reduced decomposition $s_1s_2 \cdots s_m \in W$. For q and j satisfying $1 \leq i_1 < i_2 < \cdots < i_j \leq q < i_{j+1} < \cdots < i_k \leq m$, let $s'_1s'_2 \cdots s'_q$ be another reduced decomposition of $s_1s_2 \cdots s_q$. Moreover, let $s'_{r_1}s'_{r_2} \cdots s'_{r_j} = s_{i_1}s_{i_2} \cdots s_{i_j}$ be a left subword of $s'_1s'_2 \cdots s'_q$. Then, $s'_{r_1}s'_{r_2} \cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}} \cdots s_{i_k}$ is a left subword of $s'_1s'_2 \cdots s'_qs_{q+1}s_{q+2} \cdots s_m$.*

Proof. We suppose that $s'_{r_1}s'_{r_2}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots s_{i_k}$ is not a left subword of $s'_1s'_2\cdots s'_q s_{q+1}s_{q+2}\cdots s_m$ and find a contradiction. Noting that $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a left subword of $s_1s_2\cdots s_m$, there exist natural numbers a and b such that $r_b < a < r_{b+1}$, $0 \le b \le j$, and $s'_a s'_{r_{b+1}}s'_{r_{b+2}}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots s_{i_k}$ is not a reduced decomposition. Here we put $r_0 := 0$ and $r_{j+1} := i_{j+1}$. Since $s'_{r_1}s'_{r_2}\cdots s'_{r_j}$ is a left subword of $s'_1s'_2\cdots s'_q$, $s'_a s'_{r_{b+1}}s'_{r_{b+2}}\cdots s'_{r_j}$ is a reduced decomposition. Hence, there exists a natural number p such that $j+1 \le p \le k$ and $s'_{r_{b+1}}s'_{r_{b+2}}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots s_{i_k} = s'_a s'_{r_{b+1}}s'_{r_{b+2}}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots \widehat{s_{i_p}}\cdots s_{i_k}$. So, we have

$$s'_{r_1}s'_{r_2}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots s_{i_p} = s'_{r_1}s'_{r_2}\cdots s'_{r_b}s'_a s'_{r_{b+1}}\cdots s'_{r_j}s_{i_{j+1}}s_{i_{j+2}}\cdots s_{i_{p-1}}.$$

Noting that $s'_{r_1}s'_{r_2}\cdots s'_{r_j} = s_{i_1}s_{i_2}\cdots s_{i_j}$, we get $s_{i_1}s_{i_2}\cdots s_{i_p} \le s_1s_2\cdots s_{i_p-1}$. By Lemma 1.5.(2), this contradicts that $s_{i_1}s_{i_2}\cdots s_{i_p}$ is a left subword of $s_1s_2\cdots s_m$. ∎

We define some notations.

Definition 1.7. For $x, w \in W$, we put $[x,w] := \{y \in W;\ x \le y \le w\}$, $C^-(x,w) := \{y \in [x,w];\ l(y) = l(w) - 1\}$, $c^-(x,w) := \sharp C^-(x,w)$. Note that $C^-(x,w)$ is the set of coatoms in $[x,w]$ (i.e., atoms in the dual of $[x,w]$) (cf. [10]). For a statement ST, we put $\delta(ST) := 1$ if ST is correct and $\delta(ST) := 0$ if ST is incorrect.

Then, the following combinatorial description of $p_1(x,w)$ is known (cf. [11]).

Theorem [11, Theorem A]. *For $x, w \in W$ with $x \le w$, let $s_1s_2\cdots s_m$ be a reduced decomposition of w and let $x = s_1s_2\cdots\widehat{s_{i_1}}\cdots\widehat{s_{i_2}}\cdots\widehat{s_{i_k}}\cdots s_m$ be a left subword of $s_1s_2\cdots s_m$. Then, we have*

$$p_1(x,w) = c^-(x,w) - \sum_{j=1}^{k}\delta\Big(s_{i_j}\cdots\widehat{s_{i_{j+1}}}\cdots\widehat{s_{i_{j+2}}}\cdots\widehat{s_{i_k}}\cdots s_m \not\le s_{i_j+1}s_{i_j+2}\cdots s_m\Big).$$

2. NON-NEGATIVITY

The purpose of this section is to show the following.

Theorem A. *For $x, w \in W$ with $x \le w$, let $s_1s_2\cdots s_m$ be a reduced decomposition of w and let $x = s_1s_2\cdots\widehat{s_{i_1}}\cdots\widehat{s_{i_2}}\cdots\widehat{s_{i_k}}\cdots s_m$ be a left subword of $s_1s_2\cdots s_m$. For each $j \in [k] \cup \{0\}$, we put $w_j := s_1s_2\cdots$*

$\widehat{s_{i_{j+1}}} \cdots \widehat{s_{i_{j+2}}} \cdots \widehat{s_{i_k}} \cdots s_m$. *Then, we have* $x = w_0 < w_1 < w_2 < \cdots < w_{k-1} < w_k = w$ *and* $p_1(x, w) \geq p_1(w_1, w) \geq p_1(w_2, w) \geq \cdots \geq p_1(w_{k-1}, w) \geq p_1(w, w) = 0$.

The key point for the proof of this theorem is the following.

PROPOSITION 2.1. *For* $x, w \in W$ *with* $x < w$, *let* $s_1 s_2 \cdots s_m$ *be a reduced decomposition of* w *and* $x = s_1 s_2 \cdots s_r s_{i_1} s_{i_2} \cdots s_{i_k}$ *be a left subword of* $s_1 s_2 \cdots s_m$ $(0 \leq r < r+1 < i_1, k \geq 0)$. *Put* $x_1 := s_1 s_2 \cdots s_r$, $x_2 := s_{i_1} s_{i_2} \cdots s_{i_k}$, $x' := x_1 s_{r+1} x_2$ *and* $J := S \setminus \{s_{r+1}\}$. *We suppose that* $x_1 s_{r+1} \in W^J$ *and* $s_{r+1} x_2 \nleq s_{r+2} s_{r+3} \cdots s_m$. *Then, there exists an element* y *of* W *such that* $y \in C^-(x, w) \setminus C^-(x', w)$.

Before the proof of Proposition 2.1, we will show some lemmas.

LEMMA 2.2. *Let* $x_1, x_2 \in W \setminus \{e\}$ *and* $s_1 \in S$ *and we put* $J := S \setminus \{s_1\}$. *We suppose that* $l(x_1 s_1 x_2) = l(x_1) + l(x_2) + 1$, $x_1 s_1 \in W^J$, *and there exists* $x_1' < x_1$ *and* $s_2 \in S$ *satisfying* $x_1 s_1 x_2 = x_1' s_1 x_2 s_2$. *Then, we have* $l(x_1 x_2) < l(x_1) + l(x_2)$.

We need the following to prove Lemma 2.2.

LEMMA 2.3. *We assume that Lemma* 2.2 *holds when* $l(x_2) \leq k-1$ $(k \geq 2)$. *We also suppose that* x_1 *and* x_2 *satisfy the assumption in Lemma* 2.2 *and* $l(x_2) = k$. *For such* x_2, *if there exist* $y_1, y_2 \in W$ *and* $s_3 \in S$ *such that*

$$y_2 \in W_{\{s_2, s_3\}}, \qquad y_2 s_3 < y_2, \qquad l(y_1 y_2) = l(y_1) + l(y_2), \qquad x_2 = y_1 y_2, \tag{2.1}$$

then we have $l(x_1 x_2) < l(x_1) + l(x_2)$.

Proof. Let us show this lemma by induction on $l(y_1)$. As a first step, we will consider the case that $l(y_1) = 0$. In case $y_2 = s_3(s_2 s_3)^h$ $(l(y_2) = 2h+1, h \geq 0)$, we have $x_1 s_1 s_3 (s_2 s_3)^h = x_1' s_1 s_3 (s_2 s_3)^h s_2$, using the equalities $x_1 s_1 x_2 = x_1' s_1 x_2 s_2$ and $x_2 = y_2$. So, we can easily see that there exists $z_1 < x_1' s_1 s_3$ such that $x_1 s_1 (s_3 s_2)^h = z_1 (s_2 s_3)^h s_2$. Hence, it turns out that $x_1 s_1 s_2 < x_1 s_1$. So, we can see $s_1 = s_2$ from the fact that $x_1 s_1 \in W^J$ and $J = S \setminus \{s_1\}$, and we get $x_1 s_1 s_3 (s_1 s_3)^h = x_1' s_1 s_3 (s_1 s_3)^h s_1$. Thus, we can obtain $x_1 s_3 < x_1$ and we have $l(x_1 x_2) < l(x_1) + l(x_2)$. Similarly, we can get $l(x_1 x_2) < l(x_1) + l(x_2)$ in case $y_2 = (s_2 s_3)^h$ $(l(y_2) = 2h, h \geq 1)$. We suppose that Lemma 2.3 holds when $l(y_1) \leq r-1$ and we will show Lemma 2.3 in case $l(y_1) = r$ $(r \geq 1)$. In case $y_2 = s_3(s_2 s_3)^h$ $(l(y_2) = 2h+1, h \geq 0)$, we have

$$x_1 s_1 y_1 s_3 (s_2 s_3)^h = x_1' s_1 y_1 s_3 (s_2 s_3)^h s_2. \tag{2.2}$$

Hence, we can see that $x_1 s_1 y_1 s_2 \lessdot x_1 s_1 y_1$ and we consider the following three cases: (a) $x_1 s_1 y_1 = x_1 s_1 y_1 s_2$ $(\tilde{x}_1 \lessdot x_1)$; (b) $x_1 s_1 y_1 = x_1 y_1 s_2$; (c) $x_1 s_1 y_1 = x_1 s_1 y_1' s_2$ $(y_1' \lessdot y_1)$. In the case of (a), since $l(y_1) < l(x_2)$, we get $l(x_1 y_1) < l(x_1) + l(y_1)$ and we obtain $l(x_1 x_2) < l(x_1) + l(x_2)$. In the case of (b), by (2.2), we have $x_1 y_1 s_2 s_3 (s_2 s_3)^h = x_1' y_1 s_2 s_3 (s_2 s_3)^h s_2$. Hence, it turns out that $x_1 y_1 s_3 \lessdot x_1 y_1$. Noting the equalities $y_2 = s_3 (s_2 s_3)^h$ and $x_2 = y_1 y_2$, we get $l(x_1 x_2) < l(x_1) + l(x_2)$. In the case of (c), we put $y_1'' := y_1'$ and $y_2'' := s_2 y_2$; then we can easily check that y_1'', y_2'', and s_3 satisfy the condition (2.1) and $l(y_1'') < l(y_1)$. Thus, we can use our inductive hypothesis and we have $l(x_1 x_2) < l(x_1) + l(x_2)$. Similarly, we can get $l(x_1 x_2) < l(x_1) + l(x_2)$ in case $y_2 = (s_2 s_3)^h$ $(l(y_2) = 2h,\ h \geq 1)$. We proved this lemma by induction. ∎

Proof of Lemma 2.2. Let us show this lemma by induction on $l(x_2)$. In case $l(x_2) = 1$ (we put $s_3 := x_2 \in S$), we have $x_1 s_1 s_3 = x_1' s_1 s_3 s_2$. Hence, we can easily see that there exists $z_1 \lessdot x_1' s_1 s_3$ such that $x_1 s_1 = z_1 s_2$. Noting that $x_1 s_1 \in W^J$ and $J = S \backslash \{s_1\}$, we have $s_1 = s_2$. So, we get $x_1 s_1 s_3 = x_1' s_1 s_3 s_1$. Thus, there exists $z_2 \lessdot x_1' s_1$ such that $x_1 s_1 = z_2 s_3 s_1$. Hence, it turns out that $l(x_1 s_3) < l(x_1) + l(s_3)$. We suppose that Lemma 2.2 is correct when $l(x_2) \leq k - 1$ and we will prove Lemma 2.2 in case $l(x_2) = k$ $(k \geq 2)$. Let s_3 be an element of S satisfying $x_2 s_3 \lessdot x_2$ and we put $y_1 := x_2 s_3$, $y_2 := s_3$. Then, we can easily check that y_1, y_2, and s_3 satisfy condition (2.1). Hence, by Lemma 2.3, we have $l(x_1 x_2) < l(x_1) + l(x_2)$. Thus, we proved Lemma 2.2 by induction. ∎

LEMMA 2.4. *For $x, w \in W$ with $x < w$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w and let $x = s_1 s_2 \cdots s_r s_{i_1} s_{i_2} \cdots s_{i_k}$ be a left subword of $s_1 s_2 \cdots s_m$ $(0 \leq r < r + 1 < i_1,\ k \geq 0)$. Put $x_1 := s_1 s_2 \cdots s_r$, $x_2 := s_{i_1} s_{i_2} \cdots s_{i_k}$ and $J := S \backslash \{s_{r+1}\}$. We suppose that $x_1 s_{r+1} \in W^J$ and there exists a natural number p such that $2 \leq p \leq m$ and $x_1 s_{r+1} x_2 \leq s_2 s_3 \cdots s_p$. Then, we have $s_{r+1} x_2 \leq s_{r+2} s_{r+3} \cdots s_p$.*

Proof. In case $r = 0$, this lemma is trivial. In case $r \geq 1$, we will prove this lemma by induction on k. By the inequality $x_1 s_{r+1} x_2 \leq s_2 s_3 \cdots s_p$, there exists a sequence of natural numbers $p_1, p_2, \ldots, p_{r+k+1}$ such that $2 \leq p_1 < p_2 < \cdots < p_{r+k+1} \leq p$ and

$$x_1 s_{r+1} x_2 = s_{p_1} s_{p_2} \cdots s_{p_{r+k+1}}. \tag{2.3}$$

In case $k = 0$, we have $x_1 s_{r+1} = s_{p_1} s_{p_2} \cdots s_{p_{r+1}}$. Hence, we get $s_{r+1} = s_{p_{r+1}}$. Noting the inequality $p_{r+1} \geq r + 2$, we obtain $s_{r+1} \leq s_{r+2} s_{r+3} \cdots s_{p_{r+1}} \leq s_{r+2} s_{r+3} \cdots s_p$. We suppose that Lemma 2.4 is valid for less than $k - 1$ and we will show Lemma 2.4 in the case of k $(k \geq 1)$. By (2.3), we consider the following three cases: (a) $x_1 s_{r+1} x_2 = x_1' s_{r+1} x_2 s_{p_{r+k+1}}$ $(x_1' \lessdot x_1)$; (b)

$x_1 s_{r+1} x_2 = x_1 x_2 s_{p_{r+k+1}}$; (c) $x_1 s_{r+1} x_2 = x_1 s_{r+1} x'_2 s_{p_{r+k+1}}$ ($x'_2 < x_2$). In the case of (a), by Lemma 2.2, this case contradicts the equality $l(x_1 x_2) = l(x_1) + l(x_2)$. In the case of (b), we have $x_1 x_2 = s_{p_1} s_{p_2} \cdots s_{p_{r+k}}$. Since $x_1 x_2 = s_1 s_2 \cdots s_r s_{i_1} s_{i_2} \cdots s_{i_k}$ is a left subword of $s_1 s_2 \cdots s_m$, we can see that $i_k \leq p_{r+k} < p_{r+k+1}$, as Lemma 1.5.(2). Hence, it turns out that $s_{r+1} x_2 = x_2 s_{p_{r+k+1}} \leq s_{i_1} s_{i_1+1} \cdots s_{p_{r+k+1}} \leq s_{r+2} s_{r+3} \cdots s_p$. In the case of (c), since $x'_2 < x_2$, there exists a natural number d such that $d \leq k$ and $x'_2 = s_{i_1} s_{i_2} \cdots \widehat{s_{i_d}} \cdots s_{i_k}$. By Lemma 1.5.(3), $s_1 s_2 \cdots s_r s_{i_1} s_{i_2} \cdots \widehat{s_{i_d}} \cdots s_{i_k}$ is a left subword of $s_1 s_2 \cdots s_m$. Hence, we have $s_{r+1} x'_2 \leq s_{r+2} s_{r+3} \cdots s_{p_{r+k}}$, where we use the inequality $x_1 s_{r+1} x'_2 \leq s_2 s_3 \cdots s_{p_{r+k}}$ and our inductive hypothesis. Thus, we obtain $s_{r+1} x_2 \leq s_{r+2} s_{r+3} \cdots s_{p_{r+k+1}} \leq s_{r+2} s_{r+3} \cdots s_p$. We proved Lemma 2.4 by induction. ∎

Proof of Proposition 2.1. Let us show this proposition by induction on r. In case $r = 0$, we can easily see that $s_2 s_3 \cdots s_m \in C^-(x, w) \setminus C^-(x', w)$. Let us show Proposition 2.1 in the case of r under the assumption that Proposition 2.1 is valid for less than $r - 1$ ($r \geq 1$). Put $\tilde{w} := s_1 w (= s_2 s_3 \cdots s_m)$, $\tilde{x} := s_1 x (= s_2 s_3 \cdots s_r x_2)$, and $\tilde{x}' := s_1 x' (= s_2 s_3 \cdots s_{r+1} x_2)$. Then, we can easily check that $s_2 s_3 \cdots s_r s_{i_1} s_{i_2} \cdots s_{i_k}$ is a left subword of $s_2 s_3 \cdots s_m$ and $s_2 s_3 \cdots s_{r+1} \in W^J$. Hence, by inductive hypothesis, there exists an element $\tilde{y}$ of $C^-(\tilde{x}, \tilde{w}) \setminus C^-(\tilde{x}', \tilde{w})$. We consider two cases. In case $\tilde{y} < s_1 \tilde{y}$, we can easily see $s_1 \tilde{y} \in C^-(x, w) \setminus C^-(x', w)$. In case $s_1 \tilde{y} < \tilde{y}$, noting the inequality $\tilde{y} < \tilde{w} = s_2 s_3 \cdots s_m$, there exists a natural number q such that $2 \leq q \leq m$ and $\tilde{y} = s_2 s_3 \cdots \widehat{s_q} \cdots s_m$. Since $s_1 \tilde{y} < \tilde{y}$ and $s_1 s_2 \cdots s_{q-1}$ is a reduced decomposition, there exists a natural number p such that $q + 1 \leq p \leq m$ and $\tilde{y} = s_1 s_2 \cdots \widehat{s_q} \cdots \widehat{s_p} \cdots s_m$. Hence, we have $\tilde{x} \leq s_2 s_3 \cdots \widehat{s_q} \cdots \widehat{s_p} \cdots s_m$, using $\tilde{x} \leq \tilde{y}$ and $\tilde{x} < s_1 \tilde{x}$. Therefore, we obtain $x = s_1 \tilde{x} \leq \tilde{y} < s_2 s_3 \cdots s_m$. If $x' (= x_1 s_{r+1} x_2) \leq s_2 s_3 \cdots s_m$, then, by Lemma 2.4, we get $s_{r+1} x_2 \leq s_{r+2} s_{r+3} \cdots s_m$ and this is a contradiction. Thus, we can see that $s_2 s_3 \cdots s_m \in C^-(x, w) \setminus C^-(x', w)$. It follows that we get Proposition 2.1 by induction. ∎

As a corollary of Proposition 2.1, we obtain the following.

COROLLARY 2.5. *For $x, w \in W$ with $x < w$, let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w and let $x = s_1 s_2 \cdots s_r s_{i_1} s_{i_2} \cdots s_{i_k}$ be a left subword of $s_1 s_2 \cdots s_m$ $(0 \leq r < r + 1 < i_1, k \geq 0)$. Put $x' := s_1 s_2 \cdots s_{r+1} s_{i_1} s_{i_2} \cdots s_{i_k}$ and $J := S \setminus \{s_{r+1}\}$. If $s_1 s_2 \cdots s_{r+1} \in W^J$, then we have $p_1(x, w) \geq p_1(x', w)$.*

Proof. By virtue of the theorem [11, Theorem A], we can easily see $p_1(x, w) - p_1(x', w) = c^-(x, w) - c^-(x', w) - \delta(s_{r+1} s_{i_1} s_{i_2} \cdots s_{i_k} \not\leq s_{r+2} s_{r+3} \cdots s_m)$. In case $\delta(s_{r+1} s_{i_1} s_{i_2} \cdots s_{i_k} \not\leq s_{r+2} s_{r+3} \cdots s_m) = 0$, since $c^-(x, w) \geq c^-(x', w)$ if $x' \geq x$, in general, we have $p_1(x, w) - p_1(x', w) \geq 0$.

In case $\delta(s_{r+1}s_{i_1}s_{i_2}\cdots s_{i_k} \not\leq s_{r+2}s_{r+3}\cdots s_m) = 1$, by Proposition 2.1, there exists an element y of $C^-(x,w)\setminus C^-(x',w)$. Hence, we have $c^-(x,w) > c^-(x',w)$ and we obtain $p_1(x,w) - p_1(x',w) \geq 0$. ∎

At last, we can prove Theorem A.

Proof of Theorem A. $x = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{k-1} \lessdot w_k = w$ is correct by Lemma 1.4. Since each $w_j = s_1s_2\cdots \widehat{s_{i_{j+1}}}\cdots \widehat{s_{i_{j+2}}}\cdots \widehat{s_{i_k}}\cdots s_m$ ($j \in [k]$) is a left subword of $s_1s_2\cdots s_m$, it suffices to show $p_1(x,w) \geq p_1(w_1,w)$. Note that $p_1(w,w) = 0$ by definition of the Kazhdan–Lusztig polynomials. In case $i_1 = 1$, since $s_1 \in W^{J'}$ ($J' = S\setminus\{s_1\}$), we have $p_1(x,w) \geq p_1(w_1,w)$, using Corollary 2.5. Let us consider the case that $i_1 > 1$. Put $z := s_1s_2\cdots s_{i_1}$ and $y := s_1s_2\cdots s_{i_1-1}$. For $s'_1s'_2\cdots s'_{i_1} \in W$ satisfying $s'_1s'_2\cdots s'_{i_1} = z$, we put $\rho(y, s'_1s'_2\cdots s'_{i_1}) := q$ if and only if $y = s'_1s'_2\cdots \widehat{s'_q}\cdots s'_{i_1}$ is a left subword of $s'_1s'_2\cdots s'_{i_1}$. Let $p = \min\{\rho(y, s'_1s'_2\cdots s'_{i_1});\ s'_1s'_2\cdots s'_{i_1} = z\}$ and let $\rho(y, s''_1s''_2\cdots s''_{i_1}) = p$. By Lemma 1.6, we can see that $x = s''_1s''_2\cdots \widehat{s''_p}\cdots s''_{i_1}s_{i_1+1}\cdots \widehat{s_{i_2}}\cdots \widehat{s_{i_k}}\cdots s_m$ is a left subword of $s''_1s''_2\cdots s''_{i_1}s_{i_1+1}s_{i_1+2}\cdots s_m$. Put $J := S\setminus\{s''_p\}$. Next, we will show $s''_1s''_2\cdots s''_p \in W^J$. Let $s''_1s''_2\cdots s''_p = \tilde{s}_1\tilde{s}_2\cdots \tilde{s}_a\tilde{s}_{a+1}\cdots \tilde{s}_p$, $\tilde{s}_1\tilde{s}_2\cdots \tilde{s}_a \in W^J$ and $\tilde{s}_{a+1}\tilde{s}_{a+2}\cdots \tilde{s}_p \in W_J$. If $1 \leq a < p$, noting that $\tilde{s}_{a+1}\tilde{s}_{a+2}\cdots \tilde{s}_p s''_p$ is a reduced decomposition, there exists a natural number d such that $d \leq a$ and $s''_1s''_2\cdots s''_p = \tilde{s}_1\tilde{s}_2\cdots \widehat{\tilde{s}_d}\cdots \tilde{s}_a\tilde{s}_{a+1}\cdots \tilde{s}_p s''_p$. So, we have

$$z = \tilde{s}_1\tilde{s}_2\cdots \tilde{s}_p s''_{p+1}s''_{p+2}\cdots s''_{i_1}, \qquad y = \tilde{s}_1\tilde{s}_2\cdots \widehat{\tilde{s}_d}\cdots \tilde{s}_a\tilde{s}_{a+1}\cdots \tilde{s}_p s''_{p+1}s''_{p+2}\cdots s''_{i_1}.$$

This contradicts the minimality of p. Hence, we see that $a = p$ and $s''_1s''_2\cdots s''_p \in W^J$. Thus, noting the equality $w_1 = s''_1s''_2\cdots \widehat{s''_p}\cdots s''_{i_1}s_{i_1+1}\cdots \widehat{s_{i_2}}\cdots \widehat{s_{i_k}}\cdots s_m$, we have $p_1(x,w) \geq p_1(w_1,w)$ by Proposition 2.5. Therefore, we proved Theorem A. ∎

Remark. The chain of elements $w_0 < w_1 < \cdots < w_k$ used in the proof of Theorem A appears in [2] as a unique maximal chain in $[w_0, w_k]$ with decreasing label.

We can evaluate the number of $c^-(x,w)$ for $x, w \in W$ as follows.

COROLLARY 2.6. *For $x, w \in W$ with $x \leq w$, let $s_1s_2\cdots s_m$ be a reduced decomposition and let $x = s_1s_2\cdots \widehat{s_{i_1}}\cdots \widehat{s_{i_2}}\cdots \widehat{s_{i_k}}\cdots s_m$ be a left subword of $s_1s_2\cdots s_m$. Then, we have*

$$c^-(x,w) \geq \sum_{j=1}^{k} \delta\left(s_{i_j}\cdots \widehat{s_{i_{j+1}}}\cdots \widehat{s_{i_{j+2}}}\cdots \widehat{s_{i_k}}\cdots s_m \not\leq s_{i_j+1}s_{i_j+2}\cdots s_m\right).$$

Note added in proof. After this article was accepted, we found that Matthew Dyer states the following in notes added in proof of the paper, Hecke algebras and shellings of Bruhat intervals, *Comp. Math.* **89** (1993), 91–115. "I can now show that the coefficients of q in $Q_{v,w}$ and $P_{v,w}$ are non-negative for general Coxeter systems."

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